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Integrability, analyticity, isochrony, equilibria, small oscillations, and Diophantine relations: results from the stationary Burgers hierarchy

M Bruschi[1,2], **F Calogero**[1,2] **and R Droghei**[3]

[1] Dipartimento di Fisica, Università di Roma 'La Sapienza', 00185 Roma, Italy
[2] Istituto Nazionale di Fisica Nucleare, Sezione di Roma, Italy
[3] Dipartimento di Fisica, Università Roma Tre, Italy

E-mail: mario.bruschi@roma1.infn.it, francesco.calogero@roma1.infn.it, francesco.calogero@uniroma1.it and droghei@fis.uniroma3.it

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Abstract

An *isochronous* system is introduced by modifying the *N*th ODE of the stationary Burgers hierarchy, and then, by investigating its behaviour near its equilibria, neat *Diophantine* relations are identified, involving (well-known) polynomials of arbitrary degree having *integer* zeros, or equivalently matrices the determinants of which yield such polynomials. The basic idea to arrive at such relations is not new, but the specific application reported in this paper is new, and it is likely to open the way to several analogous new findings.

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1. Introduction

The general approach to arrive at the findings reported in this paper can be described as follows (see for instance [1]). One starts from an *integrable* ODE of (*arbitrary*) order $N+1$, *all* solutions of which are *meromorphic* functions of its independent ('time') variable (the Painlevé property). One then modifies it (via an appropriate change of dependent and independent variables: see below) so that—thanks to the analyticity properties in *complex* time of the solutions of the original *integrable* ODE—the modified ODE becomes *entirely isochronous*: its solutions are *all* periodic with the same *fixed* period. (Indeed, the modification entails that the time-evolution yielded by the modified ODE corresponds essentially to the evolution yielded by the original ODE when its independent variable rotates uniformly on a circle in the *complex* plane: the *isochrony* of the solutions of the modified ODE is therefore a consequence of the *meromorphic* character of all solutions of the original ODE, considered as functions of their *complex* independent variable. The possibility to perform such a modification, transforming via this technique—as described below—an *autonomous* ODE into a modified ODE which is also *autonomous,* requires that the original system satisfy a *grading* property, which is often featured by *integrable* systems analogous to that treated herein: see [1].) One then identifies the equilibrium (i.e. time-independent) solutions of the *isochronous* ODE (in some cases this can be done explicitly: see below) and investigates, close to these equilibria, their (infinitesimally small) oscillations. The frequencies of these oscillations are given by the roots of polynomials of degree $N+1$. The fact that the ODE is *isochronous* entails that, around each equilibrium, these frequencies must *all* be *integer* multiples of a basic one. In this manner one arrives at *Diophantine* relations: *polynomials* are identified which factorize in terms of *integer* zeros.

Our route to arrive at these findings is not new, and it might appear contrived: indeed, in the context treated below, its formulation via an *isochronous* ODE could be replaced by other, equivalent approaches of a more algebraico-geometrical character (as outlined below, see remark 2 in the following section). We prefer this route because its 'physical' significance is quite transparent and its application has already yielded interesting findings (for a review see [1], including its appendix C entitled 'Diophantine findings and conjectures'). The application of this approach to the *integrable* ODE treated herein is new, hence the corresponding findings are as well new. And it is plain that analogous results are obtainable by applying the same approach to other *integrable* ODEs, this being perhaps the most interesting aspect of the findings reported below. Indeed the derivation of analogous results from the *N*-th order ODE of the KdV (rather than the Burgers) hierarchy is almost completed and will be reported soon [2].

The results of this paper are detailed in the following section 2 and proven in the subsequent section 3. A terse section 4 entitled 'Outlook' concludes the paper.

2. Results

It is well-known that the following *nonlinear* ODE of order $N+1$,

$$\left(\frac{\mathrm{d}}{\mathrm{d}\tau}\right)\left\{\frac{\mathrm{d}}{\mathrm{d}\tau}+\zeta(\tau)\right\}^{N}\cdot\zeta(\tau)=0, \tag{1}$$

is *integrable*, and in particular that *all* its solutions $\zeta(\tau)$ possess the ('Painlevé') property to be *meromorphic* functions of the independent variable τ, considered as a *complex* variable. The notation $\left\{\frac{\mathrm{d}}{\mathrm{d}\tau}+\zeta(\tau)\right\}^{N}\cdot$ indicates of course the sequential application N times of the operator $\left\{\frac{\mathrm{d}}{\mathrm{d}\tau}+\zeta(\tau)\right\}$.

The fact that the ODE (1) is *integrable* is well-known: this equation is just the *stationary* version of the *N*-th PDE of the 'Burgers' hierarchy of *integrable* PDEs (with the 'spatial' independent variable denoted here as τ).

Consistently with our approach we now replace the ODE (1) with a modified ODE, obtained via the following change of (dependent and independent) variables:

$$z(t)=\exp(\mathrm{i}\,t)\zeta(\tau),\qquad \zeta(\tau)=\exp(-\mathrm{i}\,t)z(t), \tag{2a}$$

$$\tau\equiv\tau(t)=\mathrm{i}[1-\exp(\mathrm{i}\,t)]. \tag{2b}$$

Here, and below, i is the imaginary unit, $\mathrm{i}^2=-1$. Note that the formula (2*b*) implies the relation

$$\frac{\mathrm{d}\tau(t)}{\mathrm{d}t}=\exp(\mathrm{i}\,t), \tag{3a}$$

hence,

$$\frac{\mathrm{d}}{\mathrm{d}\tau} = \exp(-\mathrm{i}\,t)\frac{\mathrm{d}}{\mathrm{d}t}. \tag{3b}$$

(Formula (2*b*) also implies $\tau(0) = 0$ hence $z(0) = \zeta(0)$; but this simple relation will play no role in the following.)

As clearly implied by the relations (2*a*) and (3*b*), our modified ODE reads

$$\left(\frac{\mathrm{d}}{\mathrm{d}t}\right)\left\{\exp(-\mathrm{i}\,t)\left[\frac{\mathrm{d}}{\mathrm{d}t} + z(t)\right]\right\}^N \cdot \exp(-\mathrm{i}\,t)z(t) = 0. \tag{4}$$

Hereafter we will consider t as the new independent *real* variable ('time'), and $z(t)$ as the new dependent variable; and it is clear (see [1] if need be) from (2) and the *meromorphic* character of the dependence of $\zeta(\tau)$ from τ, that $z(t)$ satisfies the *isochrony* property

$$z(t + 2\pi) = z(t). \tag{5}$$

A more explicit version of the ODE (4), displaying its *autonomous* character, clearly reads as follows:

$$\left[\frac{\mathrm{d}}{\mathrm{d}t} - (N+1)\mathrm{i}\right]\prod_{n=1}^{N}\left[\frac{\mathrm{d}}{\mathrm{d}t} - \mathrm{i}\,n + z(t)\right]z(t) = 0. \tag{6a}$$

This ODE is obtained from (4) by commuting the terms $\exp(-\mathrm{i}\,t)$ to the extreme left of the ODE (and finally omitting the nonvanishing factor $\exp[-\mathrm{i}\,(N+1)t]$ appearing at the extreme left of the equation); note that now the product symbol means that the operator $\frac{\mathrm{d}}{\mathrm{d}t} - \mathrm{i} + z(t)$ appears to the extreme right and the operator $\frac{\mathrm{d}}{\mathrm{d}t} - \mathrm{i}\,N + z(t)$ to the extreme left,

$$\prod_{n=1}^{N}\left[\frac{\mathrm{d}}{\mathrm{d}t} - \mathrm{i}\,n + z(t)\right] \equiv \left[\frac{\mathrm{d}}{\mathrm{d}t} - \mathrm{i}\,N + z(t)\right]\ldots\left[\frac{\mathrm{d}}{\mathrm{d}t} - \mathrm{i} + z(t)\right]. \tag{6b}$$

Consistently with our approach we now set (for infinitesimal ε),

$$z(t) = \mathrm{i}\,y + \varepsilon\exp(\mathrm{i}\,x\,t), \tag{7}$$

where clearly $z(t) = \mathrm{i}\,y$ (with y time-independent) denotes an equilibrium solution of the ODE (6*a*) and the term $\varepsilon\exp(\mathrm{i}\,xt)$ denotes the (infinitesimally small) oscillations of the solution of the ODE (6*a*) in the neighborhood of this equilibrium configuration.

The insertion of this *ansatz* in our *isochronous* ODE (6*a*) then yields, to order $\epsilon^0 = 1$, the neat formula

$$\prod_{n=0}^{N}[y - n] = 0, \tag{8a}$$

yielding the $N+1$ equilibrium configurations

$$y_k = k, \quad k = 0, 1, \ldots, N. \tag{8b}$$

Likewise, to order ε, it is easily seen that we get from (6*a*) (assuming (8) is satisfied) the equation

$$[x - (N+1)]P_N(x; y) = 0, \tag{9}$$

with

$$P_N(x; y) = \sum_{n=0}^{N}\left\{\left[\prod_{m=n+1}^{N}(x + y - m)\right]\left[\prod_{m=0}^{n-1}(y - m)\right]\right\} \tag{10a}$$

or, equivalently,

$$P_N(x;y) = \prod_{m=1}^{N}(x+y-m) + \prod_{m=0}^{N-1}(y-m) + \sum_{n=1}^{N-1}\left\{\left[\prod_{m=n+1}^{N}(x+y-m)\right]\left[\prod_{m=0}^{n-1}(y-m)\right]\right\}. \qquad (10b)$$

The equivalence among these two definitions of the polynomial $P_N(x;y)$ is consistent with the usual convention according to which a product has *unit* value if its lower limit exceeds by one unit its upper limit,

$$\prod_{\ell=m}^{n}\varphi_\ell = 1 \qquad \text{if} \quad m = n+1. \qquad (11a)$$

Actually it is convenient (see below) to adopt also the additional convention according to which

$$\prod_{j=m}^{n}\varphi_\ell = \prod_{j=n+1}^{m-1}\frac{1}{\varphi_\ell} \qquad \text{if} \quad m > n+1. \qquad (11b)$$

Note that $P_N(x;y)$ is a *monic* polynomial of degree N in x, and also a (not monic) polynomial of degree N in y; but of course the equation (9) only holds for values of y satisfying (8), and for such values of y it must have the *Diophantine* property to yield *integer* values for the N roots of the polynomial $P_N(x;y)$ (considered as a function of x). These N roots must moreover be all different among themselves and also different from the additional solution $x = N+1$ of (9)—as clearly implied by the requirement that the solution (7) of the ODE (4), of order $N+1$, satisfy the *isochrony* property (5). Hereafter we use the notation $P_N^{(k)}(x)$ for the polynomial $P_N(x;y)$ with (8*b*), and we note that it reads

$$P_N^{(k)}(x) = \sum_{n=0}^{k}\left[\frac{k!}{(k-n)!}\prod_{m=n+1}^{N}(x+k-m)\right], \qquad k = 0, 1, \ldots, N. \qquad (12)$$

The main result of this paper—proven in section 3—is the factorization formula

$$P_N^{(k)}(x) = \prod_{m=1}^{k}(x+m)\prod_{m=1}^{N-k}(x-m) = (-1)^{N-k}p_k(x)p_{N-k}(-x). \qquad (13)$$

Here and hereafter the 'shifted Pochhammer' polynomials are defined as follows:

$$p_n(x) = (x+1)_n = \prod_{m=1}^{n}(x+m), \qquad (14a)$$

of course with the convention (11*a*) entailing $p_0(x) = (x+1)_0 = 1$. In this formula and always below the 'Pochhammer' symbol $(z)_m$ has the usual meaning,

$$(z)_0 = 1,\ (z)_n = \frac{\Gamma(z+n)}{\Gamma(z)} = \prod_{\ell=0}^{n-1}(z+\ell) \qquad \text{for} \quad n = 1, 2, 3, \ldots \qquad (14b)$$

(see for instance page 56 of [3]).

The formula (13) is of course valid for $k = 0, 1, \ldots, N$, and it clearly shows that—consistently with the 'Diophantine' expectations indicated above—the *monic* polynomial $P_N^{(k)}(x)$ features the k *negative integer* zeros $-1, -2, \ldots, -k$, and the $N - k$ *positive integer* zeros $1, 2, \ldots, N - k$ (of course for $k = 0$ there are no *negative* zeros, and for $k = N$ there are no *positive* zeros).

Remarkably, by taking advantage of the convention (11*b*), this formula can be extended to read, for arbitrary *positive integer* k and *nonnegative integer* N *smaller* than k,

$$\begin{aligned}
P_N^{(k)}(x) &= \sum_{n=0}^{k}\left[\frac{k!}{(k-n)!}\prod_{m=n+1}^{N}(x+k-m)\right]\\
&= \sum_{n=0}^{N-1}\left[\frac{k!}{(k-n)!}\prod_{m=n+1}^{N}(x+k-m)\right]+\frac{k!}{(k-n)!}\\
&\quad + \sum_{n=N+1}^{k}\left[\frac{k!}{(k-n)!}\prod_{m=N+1}^{n}\frac{1}{(x+k-m)}\right]\\
&= \frac{1}{x}\prod_{m=k-N}^{k}(x+m) = \frac{p_k(x)}{x\,p_{k-N-1}(x)},\\
N &= 0, 1, \ldots, k-1, \qquad k = 1, 2, \ldots,
\end{aligned} \tag{15a}$$

and likewise, for arbitrary *nonnegative integer* k and $N = -1$,

$$\begin{aligned}
P_N^{(k)}(x) &= \sum_{n=0}^{k}\left[\frac{k!}{(k-n)!}\prod_{m=n+1}^{-1}(x+k-m)\right]\\
&= \sum_{n=0}^{k}\left[\frac{k!}{(k-n)!}\prod_{m=1}^{n}\frac{1}{(x+k-m)}\right]\\
&= \frac{1}{x}, \qquad k = 0, 1, \ldots,
\end{aligned} \tag{15b}$$

and as well, for arbitrary *nonnegative integer* k and arbitrary *negative integer* N *less* than -1,

$$\begin{aligned}
P_N^{(k)}(x) &= \sum_{n=0}^{k}\left[\frac{k!}{(k-n)!}\prod_{m=n+1}^{N}(x+k-m)\right]\\
&= \sum_{n=0}^{k}\left[\frac{k!}{(k-n)!}\prod_{m=N+1}^{n}\frac{1}{(x+k-m)}\right]\\
&= \frac{1}{x}\prod_{m=k+1}^{k-N-1}\frac{1}{(x+m)} = \frac{p_k(x)}{x\,p_{k-N-1}(x)},\\
k &= 0, 1, \ldots, \qquad N = -2, -3, \ldots.
\end{aligned} \tag{15c}$$

A related finding, also proven in section 3, is the observation that the shifted Pochhammer polynomial (14*a*) has the following determinantal representation:

$$p_k(x) = \det[c^{(k)}(x)] \tag{16}$$

with the $k \times k$ tridiagonal matrix $c^{(k)}(x)$ defined componentwise as follows:

$$c_{n,n}^{(k)}(x) = x + 2n - 1, \qquad n = 1, \ldots, k, \tag{17a}$$

$$c_{n,n+1}^{(k)}(x) = c_{n,n+1}^{(k)} = 1, \qquad n = 1, \ldots, k-1, \tag{17b}$$

$$c_{n,n-1}^{(k)}(x) = (n-1)(x+n-1), \qquad n = 2, \ldots, k, \tag{17c}$$

with all other matrix elements vanishing.

And another interesting finding, which is an immediate consequence of this result, is the observation that the shifted Pochhammer polynomials (14*a*) are as well characterized by the three-term recursion relation

$$p_n(x) = (x+2n-1)p_{n-1}(x) - (n-1)(x+n-1)p_{n-2}(x), \tag{18}$$

with the initials assignments $p_0(x) = 1$, $p_1(x) = x+1$. But of course the shifted Pochhammer polynomials are clearly characterized, see (14*a*), by the two-term recursion relation

$$p_n(x) = (x+n)p_{n-1}(x) \tag{19}$$

with $p_0(x) = 1$, and it is easily seen that this relation implies the three-term recursion relation (18) (to see this, note that (19) allows to replace, in the right-hand side of (18), the term $(x+n-1)p_{n-2}(x)$ with$p_{n-1}(x)$, thereby showing that the right-hand sides of (18) and (19) coincide).

Let us end this section with two remarks.

Remark 1. For generic values of the variables x and y the monic polynomial $P_N(x; y)$, see (10), can clearly be rewritten (using the Pochhammer notation (14*b*)) as follows:

$$P_N(x; y) = (-1)^N(-x-y+1)_N \sum_{n=0}^{N} \left[\frac{(1)_n(-y)_n}{n!(-x-y+1)_n} \right]. \tag{20}$$

The sum in the right-hand side of (20) can then be recognized as the *truncated hypergeometric function* $y_N(a, b, c, z)$ with *unit* argument, $z = 1$, upper parameters $a = 1$ and $b = -y$, and lower parameter $c = -x-y+1$, and thereby related to appropriate ${}_3F_2$ hypergeometric functions of *unit* argument (for the notation and the relevant formulas see pages 191 and 192 of [3]):

$$P_N(x; y) = (-1)^N(-x-y+1)_N \cdot \frac{(N+1)}{(N+1-y)} {}_3F_2 \left[\begin{matrix} 1, -y, -x-y+N+1; \\ -y+N+2, -x-y+1 \end{matrix} \right], \tag{21a}$$

$$P_N(x; y) = \frac{(x+y-N)_{N+1} - (y-N)_{N+1}}{x}, \tag{21b}$$

$$P_N(x; y) = (-1)^N \frac{(N+1)}{(N+1-x-y)} \frac{\Gamma(N+1-y)}{\Gamma(x+1)\Gamma(1-x-y)} \cdot {}_3F_2 \left[\begin{matrix} -x-y, -x-y+N+1, -x+1; \\ -x-y+1, -x-y+N+2 \end{matrix} \right]. \tag{21c}$$

These formulas, together with the findings reported above, see (13), clearly entail some *Diophantine* properties of certain ${}_3F_2$ hypergeometric functions with *unit* argument and appropriate parameters.

Remark 2. Analogous results to those reported here could have been obtained directly from the ODE (1), without going through its *isochronous* version (4). Then clearly the role of the equilibrium solution $z(t) = \mathrm{i}\, y$ of the *isochronous* ODE (4) would have been played by the simple single-pole solution $\zeta(\tau) = \mathrm{i}\, y(\tau - \tau_0)^{-1}$ of the ODE (1), and the role of the solution (7) would have been played by the solution $\zeta(\tau) = \mathrm{i}\, y(\tau - \tau_0)^{-1}[1 + \varepsilon(\tau - \tau_0)^x]$ (with ε infinitesimal)—the requirement that x be *integer* being then directly implied by the *meromorphic* character of all solutions of the *integrable* ODE (1).

3. Proofs

Let us first of all demonstrate our main result, namely the equality of the two expressions (12) and (13) of the monic polynomial $P_N^{(k)}(x)$. Since the polynomial (12) is clearly *monic* and of degree N (note that its highest-degree term comes from the $n = 0$ term in the sum), to prove this result it is necessary and sufficient to show that the right-hand side of (12), with $k = 0, 1, \ldots, N$, vanishes when x takes the *negative integer* values $-k, -k+1, \ldots, -1$ and the *positive integer values* $1, 2, \ldots, N-k$. Indeed in both these cases (with these *negative* respectively *positive* values of x, that should be treated separately) it is easily seen, via appropriate shifts of the summation index and the standard binomial formula

$$\sum_{q=0}^{Q}\left[\frac{(-z)^q}{q!(Q-q)!}\right] = \frac{(1-z)^Q}{Q!}, \tag{22}$$

that the polynomial (12) vanishes because it becomes proportional to the binomial $(1 - z)$ with $z = 1$ raised to a *positive* power. Our main result is thereby proven.

Let us then proceed to prove the additional finding reported in the previous section (see (16) with (17)). We start from the observation that—as it is easily seen by expanding along its last line the determinant in the right-hand side of the following formula—an alternative version of the polynomial (12) is provided by the formula

$$P_N^{(k)}(x) = \det[A^{(k)}(x)], \tag{23}$$

where $A^{(k)}(x)$ is the $(N+1) \times (N+1)$ block matrix

$$A^{(k)}(x) = \begin{pmatrix} B^{(k)}(x) & 0 \\ 0 & C^{(k)}(x) \end{pmatrix}. \tag{24}$$

The *triangular* $(N-k) \times (N-k)$ matrix $B^{(k)}(x)$ is defined componentwise as follows:

$$B_{n,n}^{(k)}(x) = x - N + k + n - 1, \qquad n = 1, \ldots, N-k, \tag{25a}$$

$$B_{n,n+1}^{(k)}(x) = B_{n,n+1}^{(k)} = 1, \qquad n = 1, \ldots, N-k-1, \tag{25b}$$

with all other matrix elements vanishing, and the $(k+1) \times (k+1)$ matrix $C^{(k)}(x)$ is defined componentwise as follows:

$$C_{n,n}^{(k)}(x) = x + n - 1, \qquad n = 1, \ldots, k, \tag{26a}$$

$$C_{n,n+1}^{(k)}(x) = C_{n,n+1}^{(k)} = 1, \qquad n = 1, \ldots, k, \tag{26b}$$

$$C_{k+1,n}^{(k)}(x) = C_{k+1,n}^{(k)} = \frac{(-1)^{k+1-n}k!}{(n-1)!}, \qquad n = 1, \ldots, k+1, \tag{26c}$$

with all other matrix elements vanishing. Recall that we are always assuming that k takes an *integer* value in the range from 0 to N.

Clearly the block structure (24) entails that an equivalent formula to (23) reads

$$P_N^{(k)}(x) = \det[B^{(k)}(x)]\det[C^{(k)}(x)], \tag{27}$$

while the triangular character of the $(N-k) \times (N-k)$ matrix $B^{(k)}(x)$, see (25), implies

$$\det[B^{(k)}(x)] = (-1)^{N-k} p_{N-k}(-x), \tag{28}$$

with the 'shifted Pochhammer' polynomial $p_{N-k}(x)$ defined by (14*a*). Hence via the formula (13) (proven above) we conclude that

$$\det[C^{(k)}(x)] = p_k(x) = (x+1)_k = \prod_{m=1}^{k}(x+m). \tag{29}$$

We now associate to the $(k+1) \times (k+1)$ matrix $C^{(k)}(x)$ another $(k+1) \times (k+1)$ matrix—clearly having the same determinant—obtained by subtracting from every column (except the last) of $C^{(k)}(x)$ its subsequent column multiplied by an appropriate coefficient, chosen so as to cancel exactly the bottom term of the resulting column. We thus get the *tridiagonal* $(k+1) \times (k+1)$ matrix $\tilde{C}^{(k)}(x)$ defined componentwise as follows:

$$\tilde{C}^{(k)}_{n,m}(x) = C^{(k)}_{n,m}(x) - C^{(k)}_{n,m+1}(x)\frac{C^{(k)}_{k+1,m}(x)}{C^{(k)}_{k+1,m+1}(x)},$$

$$n = 1, \ldots, k+1, \qquad m = 1, \ldots, k; \tag{30a}$$

$$\tilde{C}^{(k)}_{n,k+1}(x) = C^{(k)}_{n,k+1}(x), \qquad n = 1, \ldots, k+1. \tag{30b}$$

Clearly the bottom line of this matrix has all elements vanishing, except for the last one, $C^{(k)}_{k+1,k+1}(x) = C^{(k)}_{k+1,k+1}$, which equals unity, $C^{(k)}_{k+1,k+1} = 1$, see (26). We can therefore replace the formula (29) with the equivalent expression

$$\det[c^{(k)}(x)] = p_k(x) = (x+1)_k = \prod_{m=1}^{k}(x+m), \tag{31}$$

where the $k \times k$ matrix $c^{(k)}(x)$ coincides with the $(k+1) \times (k+1)$ matrix $\tilde{C}^{(k)}_{n,m}(x)$ amputated of its last line and of its last column. We thus arrive, via (26), at the formula (16) with (17), which is thereby proven.

4. Outlook

As mentioned above, we deem that the main results of our paper are the *Diophantine* factorizations (13) of the polynomial (12), as well as the related findings: the formulas (15) and the combinations of these relations with the formulas (21), entailing *Diophantine* properties of certain hypergeometric functions. As it generally happens with such findings, once such properties are uncovered it is easy to prove their validity, and there may be several alternative routes to do so. We have not found such formulas in the standard compilations of mathematical formulas where we hope they will eventually appear (especially the *Diophantine* findings involving hypergeometric functions)—but of course we cannot be quite certain that such simple and neat formulas are new. It seems to us in any case remarkable that such results can be obtained—presumably for the first time—from such a thoroughly explored class of integrable PDEs as the Burgers one.

Clearly, the approach applied in this paper to the N-th ODE of the stationary 'Burgers' hierarchy can be analogously applied to other hierarchies of *integrable* PDEs, indeed as already mentioned above analogous results for the KdV hierarchy are in the pipeline [2]. This fact is perhaps the most interesting contribution of the present paper.

Finally, let us mention that presumably an extension of some of the results reported in this paper is possible, analogous to the extension from hypergeometric to basic hypergeometric functions by Chen and Ismail [4] of the *Diophantine* findings we obtained in previous papers (see in particular [5]).

Acknowledgments

The research reported in this paper was motivated by analogous (preliminary) results for the KdV (rather than the Burgers) hierarchy, obtained by FC in March 2009 while he was a Visiting Fellow at the Isaac Newton Institute for Mathematical Studies (INI) in Cambridge in the context of the semester-programme there on Discrete Integrable Systems (DIS), and reported in a preprint deposited at INI at the end of his stay. FC wishes to thank INI and the organizers of the DIS programme for the invitation and the pleasant working atmosphere, and to acknowledge useful conversations there with A Hone, M Ismail, S P Novikov and A P Veselov on topics related to these results.

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